All-pay Auctions with Affiliated Binary Signals: Online Appendix

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This supplementary material contains the proof of Proposition 1, 3 and 6 in the main article.

A. Proof of Proposition 1

We first introduce some additional notation. Throughout the main body of this proof we assume that $v(\theta, t_i)$ depends non-trivially on θ. The other case, the affiliated private values case, is easier and dealt with at the end. When $v(\theta, t_i)$ depends on θ, i.e., when the other bidders have relevant information about bidder i's valuation, we need consider the expected payoff conditional on winning. If there is an atom at $\widehat{b}$ in the bid distribution, a bidder submitting $\widehat{b}$ ties with a positive probability for the highest bid, in which case the winner is determined by uniform rationing among tying bidders. Since the probability of getting the object depends on the number of tying bidders, we must take into account the information that the event of winning conveys about θ.

Consider the event that the state is θ_m and n (with $n = 0, \cdots, N-1$) of bidder i's opponents are tied for the highest bid $\widehat{b}$. Define $p(n, \theta_m; \widehat{b})$ as the probability of this event. The following lemma determines when a large number of bidders that tie at $\widehat{b}$ is good news and when it is bad news about θ. To state the lemma, denote the probability mass function on Θ in the presence of n tying bidders at $\widehat{b}$ by

$$q_{\widehat{b}}(\theta_m | n) := \frac{p(n, \theta_m; \widehat{b})}{\sum_{m=1}^{M} p(n, \theta_m; \widehat{b})},$$

and its cumulative distribution function by $Q_{\widehat{b}}(\theta_m | n)$. The lemma provides a simple criterion of whether $Q_{\widehat{b}}$ can be ranked in the first-order stochastic dominance: for $n' > n$, $Q_{\widehat{b}}(\theta_m | n') \leq Q_{\widehat{b}}(\theta_m | n)$ for all $\theta_m \in \Theta$. To state this criterion, let $F_*^k(\widehat{b}_-) = \lim_{b \uparrow \widehat{b}} F_*^k(b)$ denote the left-hand limit of F_*^k at $\widehat{b}$ for each type $k = L, H$. The probability of the event that type k bids at the atom $\widehat{b}$ can then be written as $\Delta^k(\widehat{b}) := F_*^k(\widehat{b}) - F_*^k(\widehat{b}_-)$.

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Lemma A.1. *For each $\theta_m \in \Theta$, the posterior distribution $Q_{\widehat{b}}(\theta_m | n)$ is*

$$\begin{Bmatrix} \text{nonincreasing in } n \\ \text{nondecreasing in } n \\ \text{independent of } n \end{Bmatrix} \quad \text{if} \quad \left(\Delta^H(\widehat{b}) - \Delta^L(\widehat{b})\right) F_*^L(\widehat{b}_-) \quad \begin{Bmatrix} > \\ < \\ = \end{Bmatrix} \quad \left(F_*^H(\widehat{b}_-) - F_*^L(\widehat{b}_-)\right) \Delta^L(\widehat{b}).$$

Proof. The proof exploits the fact that if $p(n, \theta_m ; \widehat{b})$ is log-supermodular (log-submodular) in (n, θ_m), then the posterior distribution $Q_{\widehat{b}}(\theta_m | n)$ is non-increasing (non-decreasing, respectively) in n. We hence investigate the properties of $p(n, \theta_m ; \widehat{b})$. Using our notation, the joint probability mass function is

$$\begin{aligned} p(n, \theta_m ; \widehat{b}) \quad &= \quad q(\theta) \binom{N-1}{n} \left(\alpha_m \Delta^H(\widehat{b}) + (1-\alpha_m) \Delta^L(\widehat{b})\right)^n \\ &\qquad \times \left(\alpha_m F_*^H(\widehat{b}_-) + (1-\alpha_m) F_*^L(\widehat{b}_-)\right)^{N-n-1}. \end{aligned}$$

Taking logarithms and then grouping the terms independent of θ_m into $\eta(n)$ and the terms independent of n into $\nu(\theta_m)$, we have:

$$\ln p(n, \theta_m ; \widehat{b}) \quad = \quad \eta(n) + \nu(\theta_m) + n \ln \left[\frac{\alpha_m \left(\Delta^H(\widehat{b}) - \Delta^L(\widehat{b})\right) + \Delta^L(\widehat{b})}{\alpha_m \left(F_*^H(\widehat{b}_-) - F_*^L(\widehat{b}_-)\right) + F_*^L(\widehat{b}_-)} \right].$$

Since the expression in the bracket above is strictly increasing (decreasing) in α_m if

$$\left(\Delta^H(\widehat{b}) - \Delta^L(\widehat{b})\right) F_*^L(\widehat{b}_-) \ > \ (<) \ \left(F_*^H(\widehat{b}_-) - F_*^L(\widehat{b}_-)\right) \Delta^L(\widehat{b}),$$

and since α_m is increasing in m by affiliation, the claim follows. □

Since the event of winning is more likely when the number of tying bidders n is small, winning is good news on θ whenever $q_{\widehat{b}}(\theta | n)$ is stochastically decreasing in n, and vice versa. Following this reasoning, the next lemma determines whether a small over- or under-bidding from an atom increases or decreases the payoff conditional on winning. Let $W_k(b)$ denote the expected value of the object conditional on winning with bid b and with signal $k = L, H$. We have:

Lemma A.2. *Let $\widehat{b}$ be a possible atom of at least one of the bidding distributions. If*

$$\left(\Delta^H(\widehat{b}) - \Delta^L(\widehat{b})\right) F_*^L(\widehat{b}_-) \ \geq \ \left(F_*^H(\widehat{b}_-) - F_*^L(\widehat{b}_-)\right) \Delta^L(\widehat{b}), \tag{1}$$

then we have

$$\lim_{b \downarrow \widehat{b}} W_k(b) \ \geq \ W_k(\widehat{b}) \ \geq \ \lim_{b \uparrow \widehat{b}} W_k(b). \tag{2}$$

If the inequality of (1) is reversed, then so are the inequalities of (2).

Proof. Let $V_{\widehat{b}}(n; k)$ denote the expected value of the object conditional on n other bidders bidding

$\widehat{b}$:

$$V_{\widehat{b}}(n;k) \;=\; \sum_{m=1}^{M} q_{\widehat{b}}(\theta_m \,| n\,)\, v(\theta_m, k)\,.$$

By bidding $b = \widehat{b}$, the bidder wins with probability $\frac{1}{n+1}$ if there is a tie with n other bidders. Hence, conditional on winning, the probability of tying with n other bidders is given by:

$$\frac{\frac{1}{n+1} p_{\widehat{b}}(n)}{\sum\limits_{n=0}^{N-1} \frac{1}{n+1} p_{\widehat{b}}(n)}, \quad n = 0, \cdots, N-1,$$

where $p_{\widehat{b}}(n)$ indicates the marginal probability of tying with n others at bid $\widehat{b}$. Consequently,

$$W_k(\widehat{b}) = \sum_{n=0}^{N-1} \frac{\frac{1}{n+1} p_{\widehat{b}}(n)}{\sum\limits_{n=0}^{N-1} \frac{1}{n+1} p_{\widehat{b}}(n)} V_{\widehat{b}}(n;k)\,.$$

By bidding slightly above $\widehat{b}$, the bidder wins against all bidders who pool at $\widehat{b}$, so that winning conveys no additional information on n. Conditional on winning, the probability of n bidders submitting $\widehat{b}$ is hence $p_{\widehat{b}}(n)$ and therefore

$$\lim_{b \downarrow \widehat{b}} W_k(b) = \sum_{n=0}^{N-1} p_{\widehat{b}}(n)\, V_{\widehat{b}}(n;k)\,.$$

By bidding slightly below $\widehat{b}$, a bidder wins only if there is no bidder who bids $\widehat{b}$, and hence

$$\lim_{b \uparrow \widehat{b}} W_k(b) = V_{\widehat{b}}(0;k)\,.$$

Since the probability distribution $\left(p_{\widehat{b}}(0), \cdots, p_{\widehat{b}}(N-1)\right)$ first-order stochastically dominates (strictly) the distribution $\left(\frac{p_{\widehat{b}}(0)}{\sum\limits_{n=0}^{N-1} \frac{1}{n+1} p_{\widehat{b}}(n)}, \cdots, \frac{\frac{1}{N} p_{\widehat{b}}(N-1)}{\sum\limits_{n=0}^{N-1} \frac{1}{n+1} p_{\widehat{b}}(n)}\right)$, which in turn strictly dominates the distribution $(1, 0, \cdots, 0)$, we have

$$\lim_{b \downarrow \widehat{b}} W_k(b) > (<)\, W_k(\widehat{b}) > (<) \lim_{b \uparrow \widehat{b}} W(b)$$

if $V_{\widehat{b}}(n;k)$ is strictly increasing (decreasing) in n, and

$$\lim_{b \downarrow \widehat{b}} W_k(b) = W_k(\widehat{b}) = \lim_{b \uparrow \widehat{b}} W_k(b)$$

if $V_{\widehat{b}}(n;k)$ does not depend on n. By Lemma A.1, $V_{\widehat{b}}(n;k)$ is strictly increasing (decreasing) in n if

$$\left(\Delta^H(\widehat{b}) - \Delta^L(\widehat{b})\right) F_*^L(\widehat{b}_-) > (<) \left(F_*^H(\widehat{b}_-) - F_*^L(\widehat{b}_-)\right) \Delta^L(\widehat{b})$$

and independent of n if the above inequality holds with equality, and hence the result follows. $\square$

The next lemma shows that the lowest bid in the support of the bids is made by the low-type bidders only and that it results in a payoff of zero.

Lemma A.3. *The lowest bid is $V_L(0)$ in any symmetric equilibrium and it is in the support of the low-type bidders. High-type bidders do not have an atom at $V_L(0)$. As a result, the low type earns zero in equilibrium.*

Proof. Suppose first that there is no mass point at the lowest bid $\underline{b}$. Then the probability of winning at $\underline{b}$ is zero and hence the expected payoff is also zero. It is not possible that $\underline{b} < V_L(0)$, since a slight overbidding would lead to strictly positive payoffs. It is also not possible that $\underline{b}$ is in the support of H but not L and that $\underline{b} < V_H(N-1)$ since winning at any bid $\underline{b} + \varepsilon$ would imply that all the bidders are of type H and there would be a profitable deviation for H. A bidder of type L never bids above $V_L(N-1) < V_H(N-1)$ in equilibrium. To see that it is not possible that $\underline{b}$ is in both supports, it is enough to observe that the value of the object conditional on winning is strictly higher to H than to L. Hence they cannot both earn zero expected profit.

The same argument shows that both players cannot have a mass point at $\underline{b}$. The lowest bid $\underline{b}$ cannot have a mass point for low-type bidders with $\underline{b} > V_L(0)$ since that would lead to an expected loss. Hence the claim of the lemma follows. $\square$

Lemma A.4. *Mass points are possible only at $V_L(0)$.*

Proof. Suppose that there is another mass point at some $\widehat{b} > V_L(0)$. If

$$\left(\Delta^H(\widehat{b}) - \Delta^L(\widehat{b})\right) F_*^L(\widehat{b}_-) > \left(F_*^H(\widehat{b}_-) - F_*^L(\widehat{b}_-)\right) \Delta^L(\widehat{b}),$$

then by Lemma A.2 the value of the object conditional on winning jumps upwards by bidding slightly above $\widehat{b}$. Since the probability of winning also increases by overbidding, this is a strictly profitable deviation for any bidder bidding $\widehat{b}$.

If

$$\left(\Delta^H(\widehat{b}) - \Delta^L(\widehat{b})\right) F_*^L(\widehat{b}_-) < \left(F_*^H(\widehat{b}_-) - F_*^L(\widehat{b}_-)\right) \Delta^L(\widehat{b}),$$

then $\Delta^L(\widehat{b}) > 0$ so that a low type must be bidding $\widehat{b}$ with a positive probability. By Lemma A.3, the payoff for the low type is zero, and hence the value of the object conditional on winning at $\widehat{b}$ must be zero for the low type. By Lemma A.2 a slight underbidding would increase the value conditional on winning above zero, which would then be a profitable deviation for the low type bidder.

The only case left is if

$$\left(\Delta^H(\widehat{b}) - \Delta^L(\widehat{b})\right) F_*^L(\widehat{b}_-) = \left(F_*^H(\widehat{b}_-) - F_*^L(\widehat{b}_-)\right) \Delta^L(\widehat{b}),$$

so that the expected value of the object does not depend on the number of tying bidders. Since the low type has a zero expected profit, the high type makes a strictly positive expected profit at $\widehat{b}$. But overbidding increases discretely the probability of winning without affecting the value conditional on winning, and so bidding $\widehat{b}+\varepsilon$ for ε small enough is a profitable deviation for the high type.

Obviously there cannot be a mass point at some $\widehat{b} < V_L(0)$ since overbidding would be strictly optimal for both types. □

Lemma A.5. *The support of the low-type bidders cannot have connected components of positive length.*

Proof. Suppose to the contrary that there is such a component and suppose that it is not in the support of the high-type bidder. Then winning at a higher bid implies a lower expected value and this is not compatible with the zero profit requirement in either a first-price or a second price auction.

Consider next the possibility of overlapping connected components for the two types. In the second-price auction, the bid in a symmetric equilibrium must be the value of the object conditional on tying for the winning bid (otherwise a deviation either up or down would be strictly optimal). This cannot be the same for the two types of bidders.

In the first-price auction, write the payoff of type $k = L, H$ who bids b as

$$u^{FP}(b, k|\mathbf{F}_*) = \sum_{n=0}^{N-1} p_k(n) \left(F_*^H(b)\right)^n \left(F_*^L(b)\right)^{N-n-1} (V_k(n) - b).$$

If the bidding supports overlap, then we must have

$$\frac{\partial u^{FP}(b, k|\mathbf{F}_*)}{\partial b} = 0$$

for $k = H, L$. We can write the derivative of the payoff function as:

$$\begin{aligned} \frac{\partial u^{FP}(b, k|\mathbf{F}_*)}{\partial b} &= \sum_{n=0}^{N-1} p_k(n) \left(F_*^H(b)\right)^n \left(F_*^L(b)\right)^{N-n-1} \\ &\quad \times \left[\left(n\frac{f_*^H(b)}{F_*^H(b)} + (N-n-1)\frac{f_*^L(b)}{F_*^L(b)}\right)(V_k(n) - b) - 1\right]. \end{aligned} \tag{3}$$

As a first step towards showing that the supports cannot overlap, we show that there cannot be an interval immediately above $V_L(0)$, where both types have a positive density. Let $\underline{b} \equiv V_L(0)$, and note that by the previous Lemmas we have $F_*^L(\underline{b}) > 0$ and $F_*^H(\underline{b}) = 0$. Then, evaluating (3) at $\underline{b}$, we see that all of the terms with $n \geq 2$ vanish, and we are left with

$$\begin{aligned} \left.\frac{\partial u^{FP}(b, k|\mathbf{F}_*)}{\partial b}\right|_{b=\underline{b}} &= p_k(0)\left(F_*^L(\underline{b})\right)^{N-1}\left[(N-1)\frac{f_*^L(\underline{b})}{F_*^L(\underline{b})}(V_k(0) - \underline{b}) - 1\right] \\ &\quad + p_k(1)\left(F_*^L(\underline{b})\right)^{N-2} f_*^H(\underline{b})(V_k(1) - \underline{b}) \\ &= \left(F_*^L(\underline{b})\right)^{N-2} p_k(0)\left[(N-1) f_*^L(\underline{b})(V_k(0) - \underline{b}) - F_*^L(\underline{b})\right] \end{aligned}$$

$$+\left(F_*^L\left(\underline{b}\right)\right)^{N-2} p_k\left(1\right) f_*^H\left(\underline{b}\right)\left(V_k\left(1\right)-\underline{b}\right).$$

Noting that $V_k\left(1\right) > V_k\left(0\right)$, $V_H\left(0\right) > V_L\left(0\right)$ and $\frac{p_H(1)}{p_H(0)} > \frac{p_L(1)}{p_L(0)}$, we have

$$\left.\frac{\partial u^{FP}\left(b, L|\mathbf{F}_*\right)}{\partial b}\right|_{b=\underline{b}} = 0 \quad\Longrightarrow\quad \left.\frac{\partial u^{FP}\left(b, H|\mathbf{F}_*\right)}{\partial b}\right|_{b=\underline{b}} > 0,$$

so it is not possible to have a connected component $\left(V_L\left(0\right), V_L\left(0\right)+\varepsilon\right)$ where both types are indifferent.

As a second step, we will rule out overlapping components strictly above $V_L\left(0\right)$. By usual arguments, the union of the two supports must be a connected set. Therefore, if the low type is active for $b' > V_L\left(0\right)$, there must be a region between $V_L\left(0\right)$ and b', where only the high type has a positive density. We will now show that if the high type has a positive density, the value of the low type is strictly decreasing. Since we already know that $u^{FP}(V_L(0), L|\mathbf{F}_*) = 0$, this rules out the possibility that the low type is active for any $b' > V_L\left(0\right)$.

Suppose that only the high type has a positive density at b, i.e., $f_*^H\left(b\right) > 0$ and $f_*^L\left(b\right) = 0$. Then

$$\frac{\partial u^{FP}(b, k|\mathbf{F}_*)}{\partial b} = \sum_{n=0}^{N-1} p_k\left(n\right)\left(F_*^H\left(b\right)\right)^n\left(F_*^L\left(b\right)\right)^{N-n-1}\left(\frac{nf_*^H\left(b\right)}{F_*^H\left(b\right)}\left(V_k\left(n\right)-b\right)-1\right).$$

If the high-type has a positive density, we must have

$$\frac{\partial u^{FP}(b, H|\mathbf{F}_*)}{\partial b} = \sum_{n=0}^{N-1} p_H\left(n\right)\left(F_*^H\left(b\right)\right)^n\left(F_*^L\left(b\right)\right)^{N-n-1}\left(\frac{nf_*^H\left(b\right)}{F_*^H\left(b\right)}\left(V_H\left(n\right)-b\right)-1\right) = 0.$$

Noting that $\frac{nf_*^H(b)}{F_*^H(b)}\left(V_H\left(n\right)-b\right)$ is increasing in n, we see that

$$p_H\left(n\right)\left(F_*^H\left(b\right)\right)^n\left(F_*^L\left(b\right)\right)^{N-n-1}\left(\frac{nf_*^H\left(b\right)}{F_*^H\left(b\right)}\left(V_H\left(n\right)-b\right)-1\right)$$

is single-crossing in n. Since $\frac{p_L(n)}{p_H(n)}$ is strictly decreasing in n, the single-crossing lemma implies that[1]

$$\sum_{n=0}^{N-1} \frac{p_L\left(n\right)}{p_H\left(n\right)} \cdot p_H\left(n\right)\left(F_*^H\left(b\right)\right)^n\left(F_*^L\left(b\right)\right)^{N-n-1}\left(\frac{nf_*^H\left(b\right)}{F_*^H\left(b\right)}\left(V_H\left(n\right)-b\right)-1\right) < 0.$$

Moreover, since $V_L\left(n\right) < V_H\left(n\right)$ for all n, this implies that

$$\frac{\partial u^{FP}(b, L|\mathbf{F}_*)}{\partial b} = \sum_{n=0}^{N-1} p_L\left(n\right)\left(F_*^H\left(b\right)\right)^n\left(F_*^L\left(b\right)\right)^{N-n-1}\left(\frac{nf_*^H\left(b\right)}{F_*^H\left(b\right)}\left(V_L\left(n\right)-b\right)-1\right) < 0,$$

[1] For a discrete domain N, the single-crossing lemma states that if $f : N \to \Re$ satisfies the (strict) single-crossing property and $\sum_{n\in N} f(n) = 0$, then $\sum_{n\in N} f(n)g(n) \geq (>)\ 0$ for an (strictly) increasing function $g : N \to \Re$. Note that the given properties of f imply $\sum_{n\geq k} f(n) \geq 0$ for every k. Hence the lemma follows from the fact that every increasing function can be approximated by $\sum_i \gamma_i \mathbb{1}_{\{n\geq k_i\}}$.

and hence the value of the low type must be negative for any $b > V_L(0)$. $\square$

Lemma A.6. *In a symmetric equilibrium of the second-price auction, the low-type bidders all bid $V_L(0)$ and the high-type bidders randomize using an atomless distribution on $[V_H(0), \mathbb{E}[v(\theta, t_i) | t_i = H, Y_i \geq 1]]$. In a symmetric equilibrium of the first-price auction, low-type bidders all bid $V_L(0)$ and the high-type bidders randomize using an atomless distribution on $[V_L(0), \mathbb{E}[v(\theta, t_i) | t_i = H] - p_H(0)(V_H(0) - V_L(0))]$.*

Proof. Lemmas A.1 $\sim$ A.5 imply that the low bidders must have a degenerate distribution at the lowest point and that the high-type bidders must play according to an atomless mixed strategy. The support of the high-type bidders distribution is uniquely pinned down by the constant profit condition in both auction formats. $\square$

Lemma A.6 establishes the uniqueness of a symmetric equilibrium under the assumption, maintained up to this point, that $v(\theta, t)$ depends non-trivially on θ. The case of affiliated private values, where $v(\theta, t) = v(t)$, is easier since no pay-off relevant information can be obtained by the outcome of a rationing event at a mass point. Lemma A.2 does not hold since with private valuations we must have

$$\lim_{b \downarrow \widehat{b}} W_k(b) = W_k(\widehat{b}) = \lim_{b \uparrow \widehat{b}} W_k(b)$$

for any atom $\widehat{b}$. This affects the statement of Lemma A.4, according to which no atoms above $V_L(0)$ can exist. It is easy to show that with private valuations, the unique equilibrium in the case of second-price auction involves two atoms: both types bid their own value with probability 1. The nature of the unique equilibrium in the first-price auction is unchanged.

B. Proof of Proposition 3

The result we established in Proposition 2 in the main text tells us that the unique symmetric equilibrium is monotonic if and only if

$$\frac{V_H(0)}{V_L(0)} \geq \frac{p_L(0)}{p_H(0)}. \tag{4}$$

For the proof of Proposition 3, we need therefore investigate the limiting behavior of each side of (4) as the number of bidders N increases.

Case 1 - Mineral Rights Model

We first show that the ratio $V_H(0)/V_L(0)$ on the left-hand side converges to one as $N \to \infty$ in the mineral rights model. To keep our notations simple, let $\mathbf{t} = (L, L, \cdots, L)$ denote the vector of signal realizations with $t_i = L$ for all i and $\mathbf{t}' = (H, L, \cdots, L)$ the vector with $t_i = L$ for all $i \neq 1$

and $t_1 = H$. Then the ratio can be written as

$$\frac{V_H(0)}{V_L(0)} = \frac{\mathbb{E}[v(\theta)\,|\mathbf{t}']}{\mathbb{E}[v(\theta)\,|\mathbf{t}]} = \frac{\sum_{m=1}^{M} q(\theta_m\,|\mathbf{t}')\,v(\theta_m)}{\sum_{m=1}^{M} q(\theta_m\,|\mathbf{t})\,v(\theta_m)},$$

where the posterior belief on θ given $\mathbf{t}$ can be calculated with the Bayes rule: for each θ_m,

$$q(\theta_m\,|\mathbf{t}') = \frac{q(\theta_m)\,\alpha_m\,(1-\alpha_m)^{N-1}}{\sum_{x=1}^{M} q(\theta_x)\,\alpha_x\,(1-\alpha_x)^{N-1}} \quad \text{and} \quad q(\theta_m\,|\mathbf{t}) = \frac{q(\theta_m)\,(1-\alpha_m)^{N}}{\sum_{x=1}^{M} q(\theta_x)\,(1-\alpha_x)^{N}}.$$

Since we have $\alpha_m < \alpha_{m+1}$ for each m, both posterior beliefs assign a unit mass to $\theta = \theta_1$ as $N \to \infty$. Consequently,

$$\lim_{N\to\infty} \frac{V_H(0)}{V_L(0)} = \frac{v(\theta_1)}{v(\theta_1)} = 1.$$

We next investigate the limit of the ratio $p_L(0)/p_H(0)$ as $N \to \infty$. We first put this ratio as

$$\frac{p_L(0)}{p_H(0)} = \frac{\sum_{m=1}^{M} q_L(\theta_m)(1-\alpha_m)^{N-1}}{\sum_{m=1}^{M} q_H(\theta_m)(1-\alpha_m)^{N-1}}, \tag{5}$$

where

$$q_L(\theta_m) = \frac{q(\theta_m)\,(1-\alpha_m)}{\sum_{x=1}^{M} q(\theta_x)\,(1-\alpha_x)} \quad \text{and} \quad q_H(\theta_m) = \frac{q(\theta_m)\,\alpha_m}{\sum_{x=1}^{M} q(\theta_x)\,\alpha_x}$$

are the posteriors of state θ_m after observing signal L and H, respectively. Dividing the top and bottom of (5) by $(1-\alpha_1)^{N-1}$ and then noting that $\left(\frac{1-\alpha_m}{1-\alpha_0}\right)^{N-1} \to 0$ for all $m = 1, \cdots, M-1$ as $N \to \infty$, we have

$$\lim_{N\to\infty} \frac{p_L(0)}{p_H(0)} = \frac{q_L(\theta_1)}{q_H(\theta_1)} > 1. \tag{6}$$

The first claim of Proposition 3 is then immediate from Proposition 2.

Case 2 - Affiliated Private Value Model

In the private value model, the left-hand side of (4) is simply $\frac{v_H}{v_L}$ which is constant over the number of bidders. The likelihood ratio on the other side is as in the mineral rights model, and its limit as $N \to \infty$ is given by (6) above. To complete the proof, we prove below that the ratio $\frac{p_L(0)}{p_H(0)}$ is increasing in N. To emphasize its dependence on N, we rewrite (5) as

$$\frac{p_L(0;N)}{p_H(0;N)} = \frac{\sum_{m=1}^{M} \xi_L(m)}{\sum_{m=1}^{M} \xi_H(m)},$$

where

$$\xi_k(m) = q_k(\theta_m)(1-\alpha_m)^{N-1}, \; t = L, H. \tag{7}$$

Note that the ratio

$$\frac{\xi_L(m)}{\xi_H(m)} = \frac{q_L(\theta_m)}{q_H(\theta_m)} = \frac{1-\alpha_m}{\alpha_m} \cdot \frac{\sum_{x=1}^{M} q(\theta_x)\alpha_x}{\sum_{x=1}^{M} q(\theta_x)(1-\alpha_x)}$$

is decreasing in m by affiliation.

To see how $p_L(0;N)/p_H(0;N)$ varies over N, consider next the ratio for $N+1$:

$$\frac{p_L(0;N+1)}{p_H(0;N+1)} = \frac{q_L(\theta_1)(1-\alpha_1)^N + \cdots + q_L(\theta_M)(1-\alpha_M)^N}{q_H(\theta_1)(1-\alpha_1)^N + \cdots + q_H(\theta_M)(1-\alpha_M)^N},$$

or with $\xi_k(m)$ defined in (7), we can simplify it further into

$$\frac{p_L(0;N+1)}{p_H(0;N+1)} = \frac{\sum_{m=1}^{M} \xi_L(m)(1-\alpha_m)}{\sum_{m=1}^{M} \xi_H(m)(1-\alpha_m)}.$$

The proof is done if we can show that

$$\frac{p_L(0;N+1)}{p_H(0;N+1)} > \frac{p_L(0;N)}{p_H(0;N)},$$

that is,

$$\frac{\sum_{m=1}^{M} \xi_L(m)(1-\alpha_m)}{\sum_{m=1}^{M} \xi_H(m)(1-\alpha_m)} > \frac{\sum_{m=1}^{M} \xi_L(m)}{\sum_{m=1}^{M} \xi_H(m)}. \tag{8}$$

The key here is that both $\frac{\xi_L(m)}{\xi_H(m)}$ and $(1-\alpha_m)$ are decreasing in m. The following lemma establishes (8) and hence completes the proof.

Lemma B.1. *Let M be a positive integer and $\{\delta_m\}_{m=1}^M$, $\{x_m\}_{m=1}^M$, and $\{y_m\}_{m=1}^M$ denote sequences with all strictly positive terms (i.e., $\delta_m, x_m, y_m > 0 \;\forall\; m$) such that $\delta_{m-1} > \delta_m$ and $\frac{x_{m-1}}{y_{m-1}} > \frac{x_m}{y_m}$ for all*

$m = 2, \cdots, M$. Then we have

$$\frac{\sum_{m=1}^{M} \delta_m x_m}{\sum_{m=1}^{M} \delta_m y_m} > \frac{\sum_{m=1}^{M} x_m}{\sum_{m=1}^{M} y_m}. \tag{9}$$

PROOF OF LEMMA B.1: In what follows, we will repeatedly use the fact that whenever $A, B, a, b > 0$ and $A/a > B/b$, we have

$$\frac{Aq + B}{aq + b} > \frac{A + B}{a + b} \tag{10}$$

for $q > 1$ (this is easy to prove by differentiating the left-hand side with respect to q).

We prove the lemma using induction. First, (9) is clearly true if $M = 2$: If $\delta_1 > \delta_2$ and $\frac{x_1}{y_1} > \frac{x_2}{y_2}$, we have

$$\frac{\delta_1 x_1 + \delta_2 x_1}{\delta_1 y_1 + \delta_2 y_1} = \frac{\frac{\delta_1}{\delta_2} x_1 + x_2}{\frac{\delta_1}{\delta_2} y_1 + y_2} > \frac{x_1 + x_2}{y_1 + y_2},$$

where the inequality uses (10).

Fix an integer $M > 2$. As an induction hypothesis, suppose that (9) holds when the summation is taken from $m = 2$ to $m = M$:

$$\frac{\sum_{m=2}^{M} \delta_m x_m}{\sum_{m=2}^{M} \delta_m y_m} > \frac{\sum_{m=2}^{M} x_m}{\sum_{m=2}^{M} y_m},$$

whenever $\delta_{m-1} > \delta_m$ and $\frac{x_{m-1}}{y_{m-1}} > \frac{x_m}{y_m}$ for all $m = 3, \cdots, M$. Then, taking the summation from $m = 1$, we can write

$$\frac{\sum_{m=1}^{M} \delta_m x_m}{\sum_{m=1}^{M} \delta_m y_m} = \frac{\delta_1 x_1 + \delta_2 \left(x_2 + \frac{\delta_3}{\delta_2} x_3 + \cdots + \frac{\delta_M}{\delta_2} x_M \right)}{\delta_1 y_1 + \delta_2 \left(y_2 + \frac{\delta_3}{\delta_2} y_3 + \cdots + \frac{\delta_M}{\delta_2} y_M \right)}. \tag{11}$$

Let

$$\chi := \frac{x_2 + x_3 + \cdots + x_M}{x_2 + \frac{\delta_3}{\delta_2} x_3 + \cdots + \frac{\delta_M}{\delta_2} x_M}. \tag{12}$$

Since $\frac{\delta_k}{\delta_2} < 1$ for all $k = 3, ..., M$, we have $\chi > 1$. Using this defitition, we can write the term in the parantesis in the nominator of (11) as:

$$x_2 + \frac{\delta_3}{\delta_2} x_3 + \cdots + \frac{\delta_M}{\delta_2} x_M = \frac{1}{\chi} (x_2 + x_3 + \cdots + x_M). \tag{13}$$

Since $\left(1, \frac{\delta_3}{\delta_2}, \frac{\delta_4}{\delta_2}, \cdots, \frac{\delta_M}{\delta_2}\right)$ is a decreasing sequence, the induction hypothesis gives:

$$\frac{x_2 + \frac{\delta_3}{\delta_2} x_3 + \cdots + \frac{\delta_M}{\delta_2} x_M}{y_2 + \frac{\delta_3}{\delta_2} y_3 + \cdots + \frac{\delta_M}{\delta_2} y_M} > \frac{x_2 + x_3 + \cdots + x_M}{y_2 + y_3 + \cdots + y_M},$$

which we can rearrange as

$$y_2 + \frac{\delta_3}{\delta_2} y_3 + \cdots + \frac{\delta_M}{\delta_2} y_M \;<\; \frac{x_2 + \frac{\delta_3}{\delta_2} x_3 + \cdots + \frac{\delta_M}{\delta_2} x_M}{x_2 + x_3 + \cdots + x_M} y_2 + y_3 + \cdots + y_M$$
$$= \frac{1}{\chi} \left(y_2 + y_3 + \cdots + y_M \right), \tag{14}$$

where the last equality uses (12). Plugging equality (13) and inequality (14) in (11) gives

$$\frac{\sum_{m=1}^{M} \delta_m x_m}{\sum_{m=1}^{M} \delta_m y_m} > \frac{\delta_1 x_1 + \frac{\delta_2}{\chi} \left(x_2 + x_3 + \cdots + x_M \right)}{\delta_1 y_1 + \frac{\delta_2}{\chi} \left(y_2 + y_3 + \cdots + y_M \right)}$$
$$= \frac{\frac{\delta_1 \chi}{\delta_2} x_1 + x_2 + \cdots + x_M}{\frac{\delta_1 \chi}{\delta_2} y_1 + y_2 + \cdots + y_M} > \frac{\sum_{m=1}^{M} x_m}{\sum_{m=1}^{M} y_m},$$

where the last inequality uses (10) and the facts that $\frac{\delta_1 \chi}{\delta_2} > 1$ (since $\delta_1 > \delta_2$ and $\chi > 1$) and that $\frac{x_1}{y_1} > \frac{x_2 + \cdots + x_M}{y_2 + \cdots + y_M}$ (since $\frac{x_1}{y_1} > \frac{x_m}{y_m}$ for all $m = 2, \cdots, M$). $\square$

C. Proof of Proposition 6

If $v_H q_H(\theta_1) \geq v_L q_L(\theta_1)$, then by Propositions 1 and 3 in the main text both standard and all-pay auctions result in an efficient allocation for all N. Therefore, the first item of Proposition 6 follows from Lemma 3, which states that the bidder rent is higher in the standard auctions than in the all-pay auction.

If $v_H q_H(\theta_1) < v_L q_L(\theta_1)$, then by Propositions 2 and 3 the equilibrium is non-monotonic in the all-pay auction for large N, in particular, the equilibrium bidding distributions for both types of players are intervals containing 0. To prove the second item of Proposition 6, we need to show that there is a bid $b' > 0$ such that if only high types bid above b' whenever $N \geq N'$ for some $N' < \infty$, then there exists $\delta > 0$ such that by bidding b' a low type earns an expected payoff of at least δ. In this case, we can conclude that there exists an $\varepsilon > 0$ such that $\Pr\{\widetilde{b}_N^L > b'\} > \varepsilon$, where $\widetilde{b}_N^L$ is the maximal bid by a low type bidder in a game with N bidders. Since also $\Pr\{\widetilde{b}_N^H < b'\} \geq \frac{b'}{v_H}$, there is a strictly positive probability that the low type wins, and the claim follows.

Any $b' < v_H$ is in $\mathrm{supp}[F_{*,N}^H]$ for N large enough. Hence for such b', we have $U_{*,N}^H = 0$. Suppose that only high types bid above b'. Then we have

$$\sum_{m=1}^{M} q_H(\theta_m) \, \mathbb{E}\pi_{m,N}(b') \;=\; \frac{b'}{v_H}, \tag{15}$$

where $\pi_{m,N}(b')$ is the (random) probability of winning with bid b'in state m if there are N bidders, i.e.

$$\pi_{m,N}\left(b'\right) \;=\; F^H_{*,N}\left(b'\right)^{N^H_{m,N}},$$

where $N^H_{m,N}$ is the (random) number of high types in state m if the total number of bidders is N. By the law of large numbers, $\frac{N^H_{m,N}}{N} \to \alpha_m$ almost surely. Hence

$$\pi_{m,N}\left(b'\right) \;\to\; F^H_{*,N}\left(b'\right)^{\alpha_m N}$$

in probability. It follows from this that

$$\pi_{m,N}\left(b'\right) \;\to\; \left(\pi_{1,N}\left(b'\right)\right)^{\frac{\alpha_m}{\alpha_1}}$$

and therefore, noting that $\lim_{b\to 0}\pi_{m,N}\left(b\right)=0$ for all m, and $\alpha_1<\alpha_m$ for $m>1$, we have

$$\lim_{k\to\infty}\lim_{N\to\infty}\frac{\pi_{m,N}\left(b'_k\right)}{\pi_{1,N}\left(b'_k\right)}=0, \tag{16}$$

where $\{b_k\}_{k=1}^{\infty}$ is a sequence with $b'_k\to 0$. Combining (15) and (16), we have

$$\lim_{k\to\infty}\lim_{N\to\infty} v_H q_H\left(\theta_1\right)\frac{\pi_{1,N}\left(b'_k\right)}{b'_k} \;=\; 1,$$

which, along with our assumption $v_H q_H\left(\theta_1\right) < v_L q_L\left(\theta_1\right)$, implies that

$$\lim_{k\to\infty}\lim_{N\to\infty} v_L q_L\left(\theta_1\right)\frac{\pi_{1,N}\left(b'_k\right)}{b'_k} \;=\; \frac{v_L q_L\left(\theta_1\right)}{v_H q_H\left(\theta_1\right)} \;>\; 1.$$

The inequality means that for small enough b', low types get a strictly positive payoff. This contradicts the fact that low type must obtain a payoff of zero in equilibrium. It follows that $\lim_{k\to\infty}\lim_{N\to\infty}\Pr\{\widetilde{b}^L_N > b'_k\} > 0$. Noting that for any $b'>0$, $\Pr\{\widetilde{b}^H_N < b'\} \geq \frac{b'}{v_H}$, we then note that the probability that the low type wins the auction is bounded away from zero. $\square$